Algebras of Deductions in Category Theory[*]

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Abstract

In proof theory we use category theory to obtain identity criteria for deductions. Inference rules, by which we pass from one deduction to another, correspond there to partial algebraic operations. This is an algebra built on deductions and not on formulae. Logic is thereby tied to algebras of another kind than Boolean algebras, Heyting algebras, and similar lattice algebras investigated in universal algebra, induced by equivalence between formulae. Algebras of deductions are based on categories with structure given by so general and so important mathematical notions like product, coproduct, exponent, tensor product,... Here logic does not formalize other areas of mathematics to investigate them by its own means, but the subject matter of logic itself coincides with something investigated in contemporary algebra and other related areas of mathematics.

1 Logical models from universal algebra

Algebras associated with logic are first of all Boolean algebras. The semantics of classical propositional logic is built upon the two-element Boolean algebra, the two elements being interpreted as *truth* and *falsity*. The models of this logic are valuations into the two-element Boolean algebra, i.e. homomorphisms from the absolutely free algebra of formulae into the two-element Boolean algebra such that Boolean functions correspond to connectives. When the algebra of propositional formulae is factored through the equivalence relation induced by equivalence of formulae, i.e. identity of truth-value for every valuation, one obtains the Lindenbaum algebra of classical propositional logic, which is a freely generated Boolean algebra, with as much free generators as there are propositional letters.

The connection between Boolean algebras and classical logic is exhibited on a less technical level by the explanation that is given of the classical connectives of conjunction, disjunction and negation via intersection, union and complement. These are the operations of a set Boolean algebra, i.e. a Boolean algebra whose elements are sets. One encounters this explanation nowadays at the earliest stages

of schooling, usually accompanied by diagrams of Euler and Venn. It is however questionable how much the notion of intersection, which is defined in terms of conjunction, can serve to explain conjunction. Conjunction is the more basic notion, and not intersection, and analogously for the notion of disjunction versus union, and the notion of negation versus complement.

With intuitionistic logic the role of Boolean algebras is taken over to a great extent by Heyting algebras, which make a more general class of algebras. These are distributive lattices with a least element and a binary operation of *residual* (also called *relative pseudo-complement*) with respect to the binary meet operation. This means that $c \cap a \leq b$ iff $c \leq a \Rightarrow b$, where $\cap$ is meet and $\Rightarrow$ is residual. Boolean algebras are complemented distributive lattices, where the residual $a \Rightarrow b$ is $-a \cup b$, with $-$ being the unary operation of complement and $\cup$ being join. The operation of residual in Heyting algebras, as well as in Boolean algebras, which are particular Heyting algebras, corresponds to the connective of implication.

The Lindenbaum algebra of intuitionistic propositional logic is a freely generated Heyting algebra, and the Kripke models of this logic, the most standard models one encounters for it, are obtained through a representation of Heyting algebras in preorderings. Heyting algebras are closely tied to topological Boolean algebras, i.e. Boolean algebras with unary operations corresponding to interior or closure. Topological Boolean algebras give models of the modal logic S4.

Various other logics have algebraic models of a similar kind, models out of the realm of universal algebra that resemble Boolean and Heyting algebras. In particular, in all these models the elements of the algebra are related to formulae, which are of the grammatical category of *propositions*. These elements are the values formulae may take. In the case of classical propositional logic and the two-element Boolean algebra, the values are the two truth-values. In other cases the values are more unusual, but they are again assigned to formulae, i.e. propositions.

2   Deductions make a category

Matters become different if in the algebraic modelling of logics the elements of the algebra are not expected any more to be assigned to formulae, i.e. propositions. For that one has to make a shift in the semantic conception of logic and language. One may cease to consider propositions as the main unit of language. One may look for a wider context in which propositions partake, and when logic is our inspiration that wider context is naturally found in *deductions*. The term *deduction*, which will later become for us a technical term, is intuitively synonymous with *inference*, or *proof from hypotheses*.

For the time being, to simplify matters we shall speak of deductions where we have not more than a single premise. This is not an essential restriction if we assume as usual that in deductions we have only finitely many premises, and we assume moreover that we have a connective like conjunction to connect these finitely many premises. The empty conjunction, i.e. a propositional constant like

$\top$, replaces the empty set of premises. So we assume that in deductions we have a single premise and a single conclusion.

Then one may ask whether the deduction from premise A to conclusion B is completely determined by giving A and B. Is this deduction just the ordered pair (A, B)? Many logicians, if they wouldn't give to this question an explicitly affirmative answer, behave as if they would. They believe that consequence *relations* fully explain deductions. We may deduce B from A if and only if B is a consequence of A, i.e., if and only if the ordered pair (A, B) belongs to the consequence relation tied to our logic. Either there is a deduction from A to B, or there is none. There cannot be more than one deduction with the same premise and the same conclusion.

An alternative is that there may be more than one deduction with the same premise and the same conclusion. Suppose A is the conjunction $B \wedge B$. Then the deduction from $B \wedge B$ to B obtained by applying the first-projection rule *from $C \wedge D$ deduce C* and the deduction from $B \wedge B$ to B obtained by applying the second-projection rule *from $C \wedge D$ deduce D* would not be the same deduction. In another example, one may deduce $B \wedge B$ from $B \wedge B$ either by applying the identity rule *from E deduce E* or the commutativity rule *from $C \wedge D$ deduce $D \wedge C$*. This gives two different deductions. Still another example is given by the deductions from $B \wedge (B \Rightarrow B)$ to B obtained by applying either the first-projection rule or the rule *from $C \wedge (C \Rightarrow D)$ deduce D*, which is based on modus ponens.

In this alternative perspective deductions do not make a binary relation, i.e. a set of ordered pairs, but a graph where for the same ordered pair (A, B) there may be more than one arrow joining A and B. Deductions should be closed under composition and for every pair (A, A) there should be an identity arrow, i.e. identity deduction, from A to A, based on the identity rule. If we assume that composition is associative and that composing a deduction with the identity deduction is equal to this deduction, we obtain a category. In this category the formulae, i.e. propositions, are the objects and the deductions are the arrows. From a categorial perspective these arrows are more important.

3   Conjunction and product

In the categorial perspective we do not have only operations on objects, but also operations on arrows. These operations may be only partial, as composition is. The algebraic structure brought by these operations on objects and on arrows tied to connectives and other logical constants is of a different kind than what we had with Boolean and Heyting algebras. Before the binary connective of conjunction corresponded to lattice meet, and now it will correspond to the bi-endofunctor of product. Meet corresponds set-theoretically to intersection, while product corresponds to another, even more important, set-theoretic operation— namely, Cartesian product. Product on objects in the category **Set** of sets with

functions as arrows is Cartesian product.

Tied to product, we have for every pair of formulae A_1 and A_2, the projection arrows $p^1_{A_1,A_2} \colon A_1 \wedge A_2 \to A_1$ and $p^2_{A_1,A_2} \colon A_1 \wedge A_2 \to A_2$, which correspond to the first-projection and second-projection rules of deduction mentioned above. We have moreover the partial operation of pairing of arrows, which applied to the arrows $f_1 \colon C \to A_1$ and $f_2 \colon C \to A_2$ gives the arrow $\langle f_1, f_2 \rangle \colon C \to A_1 \wedge A_2$. Pairing corresponds to the natural deduction rule of conjunction introduction. For the arrow $h \colon C \to A_1 \wedge A_2$ and for $\circ$ being composition, we have the equations

$$p^1_{A_1,A_2} \circ \langle f_1, f_2 \rangle = f_1, \quad p^2_{A_1,A_2} \circ \langle f_1, f_2 \rangle = f_2, \quad \langle p^1_{A_1,A_2} \circ h, p^1_{A_1,A_2} \circ h \rangle = h.$$

These equations categorially define product. Replace $\wedge$ by $\times$ and, for $g_1 \colon A_1 \to B_1$ and $g_2 \colon A_2 \to B_2$, define the total operation $\times$ on arrows by

$$g_1 \times g_2 =_{df} \langle g_1 \circ p^1_{A_1,A_2}, g_2 \circ p^2_{A_1,A_2} \rangle \colon A_1 \times A_2 \to B_1 \times B_2.$$

The operation $\times$ on objects and the operation $\times$ on arrows give the biendofunctor of product.

The equations assumed above make however perfect logical sense. They correspond to reductions involved in normalization in natural deduction (the first two equations correspond to β-reduction and the third to η-reduction). So it is natural to assume these equations as equations between deductions.

A category is a preorder when for every pair of objects A and B we have at most one arrow $f \colon A \to B$. Product becomes meet when our category is not only a preorder, but also a partial order, which means that if we have a pair of arrows $f \colon A \to B$ and $g \colon B \to A$, then the objects A and B must be the same. In a preorder, the equations assumed for product are trivially satisfied.

This categorial characterization of conjunction applies both to classical conjunction and intuitionistic conjunction. The connective of conjunction is the same in both logics. In the old universal-algebraic characterization too, conjunction was characterized in both logics as a meet operation.

4   Conjunction and disjunction

Analogously, the binary connective of disjunction, which before corresponded to join and union, now corresponds to binary coproduct, which set-theoretically is disjoint union on objects. This applies again both to classical and intuitionistic disjunction. It is often said that intuitionists understand disjunction not as it is understood in classical logic, and that because of that excluded middle fails for them. This should be taken carefully, because in a context where disjunction is alone—this is a context in which we do not have theorems, but we do have deductions—classical and intuitionistic disjunction do not differ. The essential novelty of intuitionistic logic is a new connective of implication, and not disjunction. This implication underlies negation, which we find in excluded middle (see below, the second paragraph of the next section).

When the connectives of conjunction and disjunction are together, then the universal-algebraic characterization is given by distributive lattices, and again classical and intuitionistic logic do not differ. Matters may however become different in terms of algebras of deductions.

The categories for intuitionistic conjunctive-disjunctive deductions are categories with binary (and hence all nonempty finite) products and coproducts, with product isomorphically distributive over coproduct; namely, there would be a natural isomorphism of the type $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$. There are also natural transformations of the dual type $A \vee (B \wedge C) \rightarrow (A \vee B) \wedge (A \vee C)$, and of its converse type $(A \vee B) \wedge (A \vee C) \rightarrow A \vee (B \wedge C)$, but they need not be isomorphisms. This is an asymmetry we have also in the category **Set**, where Cartesian product is isomorphically distributive over disjoint union, but not vice versa. When we restrict ourselves to finite sets, this reduces to the following equation for natural numbers $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, without the dual equation $a + (b \cdot c) = (a + b) \cdot (a + c)$ being correct.

We may add to these categories also the empty product and coproduct, which are the terminal object $\top$ and initial object $\bot$. (An object is terminal when for every other object to it there is a single arrow; dually, it is initial when from it to every other object there is a single arrow.) These two objects correspond to the propositional constants $\top$ and $\bot$, which in the universal-algebraic approach correspond to the least element and greatest element of our lattices—an assumption made both for classical logic and intuitionistic logic. The assumptions about the terminality of $\top$ and the initiality of $\bot$ are parallel to the assumptions of binary product for $\wedge$ and binary coproduct for $\vee$. They are the same in classical and intuitionistic logic.

The isomorphic distributivity of conjunction over disjunction in intuitionistic logic is a consequence of the assumption that the monoendofunctor $A \wedge$ has as a right-adjoint $A \Rightarrow$, which involves intuitionistic implication $\Rightarrow$. This means that we have a natural bijection between the hom sets $Hom(A \wedge B, C)$ and $Hom(B, A \Rightarrow C)$. Hence there is a one-to-one correspondence between deductions of the type $A \wedge B \rightarrow C$ and deductions of the type $B \rightarrow A \Rightarrow C$. From left to right this correspondence is tied to the deduction theorem, and from right to left it has to do with modus ponens; from the identity deduction of type $A \Rightarrow C \rightarrow A \Rightarrow C$ one passes to the modus ponens deduction of the type $A \wedge (A \Rightarrow C) \rightarrow C$. (The naturality involved in this correspondence is given by transformations natural in B and C between the functors into the category **Set** involved in $Hom(A \wedge B, C)$ and $Hom(B, A \Rightarrow C)$; see [11], Sections II.2 and IV.1.)

With this adjunction involving $A \wedge$ and $A \Rightarrow$, together with the assumption that we have all finite products (including the empty one), we obtain *cartesian closed categories*; if we assume moreover that we have all finite coproducts including the empty one), we obtain *bicartesian closed categories* (see [10], Sections I.3 and I.8). The category **Set** is a bicartesian closed category.

5   Categories of classical deductions

In classical algebras of deductions there are arrows of the types $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$ and $A \vee (B \wedge C) \to (A \vee B) \wedge (A \vee C)$, as well as of the converse types, but Boolean duality would make us expect that either both of these kinds of arrows are isomorphisms or neither is. It is not in the Boolean spirit that $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$ is an isomorphism while $A \vee (B \wedge C) \to (A \vee B) \wedge (A \vee C)$ is not. The asymmetry of the bicartesian closed category **Set** that we have with respect to distribution of Cartesian product over disjoint union is foreign to the Boolean spirit. Since this asymmetry is a consequence of the adjunction involving $A \wedge$ and $A \Rightarrow$, we should not expect this adjunction for classical logic.

If $\neg A$ is defined as $A \Rightarrow \bot$, it may seem natural to assume in classical logic that A is isomorphic to $\neg\neg A$, or that there is at least a natural transformation whose components are of the type $\neg\neg A \to A$. There is an argument that shows that every bicartesian closed category together with one of these assumptions, or a similar assumption, is a preorder (see [2], Section 5, or [4], Section 14.3). It was concluded from that there cannot be any interesting categorial proof theory for classical logic, i.e. that there is no interesting categorification of Boolean algebras. But why must we assume that we have to start with a bicartesian closed category?

The book [4] is a detailed attempt to build a plausible categorial theory of classical deductions. The centre-piece of that book is a notion of category with finite products and coproducts, which distribute one over the other, but neither of these distributions need be an isomorphism. This way, in the Boolean spirit, symmetry is kept. For this notion of category it is established that it is coherent in the following sense. There is faithful functor from such a category freely generated by a set of objects to the category of relations between finite ordinals. This coherence result is of a semantical kind. It permits us to decide whether two arrows are equal by checking whether two simple graphs that come from the relations associated with the arrows are equal. This is a result of a semantical kind analogous to the usual completeness theorems of logic, which enable us to decide provability by looking into a manageable model.

For this result there is given in [4] a long and rather involved proof based on a categorial version of Gentzen's cut elimination theorem for conjunctive-disjunctive classical logic, which is based on plural, i.e. multiple-conclusion, sequents. It is not established only that for every deduction we have a cut-free deduction with the same premise and conclusion, but it is established moreover that the two deductions are identical: the arrows standing for them are equal as arrows in our categories. It is remarkable that this equality of arrows has both a semantical justification via coherence, and a syntactical one via the cut-free normal from, and that the two justifications agree. Both justifications respect the Boolean duality between conjunction and disjunction. In the semantical justification product and coproduct are modeled by the same biproduct, while in the syntactical justification plural sequents treat conjunction and disjunction in the same way. All this makes

quite plausible, and mathematically interesting, the proposed notion of identity of deduction for classical propositional logic.

This categorification which covers the conjunctive-disjunctive core of classical logic, together with the propositional constants $\top$ and $\bot$, is then extended with negation, and hence also implication, so that we do not have the adjunction involving $A \wedge$ and $A \Rightarrow$. We relinquish this adjunction of intuitionistic logic, and hence classical implication will not mirror deductions, as intuitionistic implication does. The resulting nontrivial categories are related to linear algebra.

The relationship with linear algebra is brought by having an operation of union, or addition, of deductions, and a zero deduction. In terms of relations, union of deductions is union of relations, the zero deduction is the empty relation, and union with the empty relation leaves a relation unchanged. Composition of relations corresponds to multiplication of zero-one matrices and union of relations corresponds to matrix addition. In the presence of zero deductions, we obtain a unique normal form for deductions like in linear algebra: every matrix is the sum of matrices with a single 1 entry, the others being zero. This normal form is related to cut elimination, i.e. composition elimination (see [4], Sections 1.6 and 14.4, Chapters 13 and 14).

6   Categories of deductions in other logics

The book [5] deals with categories of deductions in classical linear logic, presumably the main substructural logic, and an important alternative logic. In linear logic we have a binary connective of so-called *multiplicative* conjunction, which in terms of the algebra of deductions corresponds to tensor product. For this connective we assume associativity and commutativity, but not what corresponds to the structural rules of contraction and thinning. The algebras of deductions here come from categories to which belongs the category of vector spaces over a fixed field with linear transformations as arrows. Categories of deductions in intuitionistic linear logic are given by symmetric monoidal closed categories (see [11], Section VII.7), for which a coherence theorem is established in [9]. (This coherence theorem should be rephrased as a result taking account of syntax, as in [4] and [5].) The coherence proved in [5] is of the same kind, and it is established with respect to relations that one finds in Brauer algebras (for references see [5]). Brauer algebras, which come from representation theory, are related to Temperley-Lieb algebras, which came to play an important role in knot theory after Jones' seminal results in this field. Temperley-Lieb algebras are also related to a coherence result for adjunction (see [1], [3] and references therein).

The relations one finds in Brauer algebras are a particular case of relations one finds in a category called **Gen**, which as arrows has relations involved in the generality of deductions, i.e. relations connecting letters in a deduction that must remain the same in all generalizations of this deduction based on the same rules (see Section 2 above). The category **Gen** and the category of *split preorders*,

of which it is a subcategory, are investigated in detail in [7]. These categories have properties relating them to Frobenius algebras, which have recently played a prominent role in two-dimensional topological quantum field theories—an area that seems far removed from logic and deduction theory. These matters (see [8], and references mentioned therein) are related to the coherence result for adjunction, and to Temperley-Lieb algebras, which we mentioned above. Frobenius algebras are also related to a coherence result involving deductions in the modal logic S5 (see [6]).

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